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Operation Properties and Algebraic Application of Covering Rough Sets

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Abstract. Rough set theory is one of the most important tools for data mining. The covering rough set (CRS) model is an excellent generalization of Pawlak rough sets. In this paper, we first investigate a number of basic properties of two types of CRS models. Especially, we study the operation properties of the two types of CRS models with respect to the unary covering. Meanwhile, several corresponding algorithms are constructed for computing the intersection and union of rough sets and some examples are employed to illustrate the effectiveness of these algorithms.Finally, as an application of the operation properties of CRS, some basic algebraic properties of CRS are explored. It is evident that these results will enrich the theory of covering rough sets.

Keywords: Rough sets, unary, covering, operation properties, algebraic properties

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1. Introduction

Rough set theory, initiated by Pawlak [37, 38], is an excellent tool to handle vagueness and uncertainty in data analysis. It is an efficient method employed in many areas: uncertainty reasoning [11, 43], rule extraction [1, 53], feature selection [7, 14, 15, 26, 29, 52], granular computing [3, 27, 31, 41, 42, 58, 59, 62], knowledge reduction [22-24, 28, 46], and others [6, 8, 10, 54, 60, 68, 70, 73].

It is obvious that an equivalence relation or a partition of the universe plays an important role in Pawlak rough set model. However, the requirement of an equivalence relation is a very restrictive condition for many real-world applications. To overcome the limitation, many researchers have generalized the Pawlak rough set theory. Several meaningful and interesting extensions of equivalence relation have been proposed, such as similarity relation [51, 66], tolerance relation [50, 57, 66], neighborhood systems [62], fuzzy systems [55, 56]. In 1983, Zakowski first proposed the notion of covering based on rough set approximations [69]. A pair of lower and upper approximation operators are defined by a straightforward generalization of the Pawlak definition. Since then, a great number of diversity lower and upper approximation operators have been proposed [2, 5, 13, 32-35, 45, 49, 63, 69, 72, 74-78]. Yao studied dual approximation operators by using coverings produced by the predecessor and/or successor neighborhoods of serial or inverse serial binary relations [64, 65]. By modifying Zakowski's definition, Pomykala investigated two pairs of dual approximation operators and studied many properties of covering rough sets based on tolerance relations [44]. In addition, Zhu et al. researched six types of approximation operators and investigated their properties. Furthermore, the relationships of them have been discussed [74-78]. Mordeson examined the pair of Zakowski approximation operators by considering semi-reduced covering [36]. From the above, we can see that many excellent results of CRS theory have been proposed. These results enrich and extend the applications of rough set theory.

Since Pawlak proposed the theory of rough sets in 1982, the researches of operation properties and algebraic properties on rough sets have been started. An algebraic approach to rough set theory was first presented by Iwiński in 1987 [16]. Since then, substantial conclusions on operation properties and algebraic properties of rough sets have been done [4, 9, 12, 17-21, 39, 40, 48, 66, 67, 71]. However, it is still an open problem regarding the operation properties of CRS. Therefore, the main objective of this paper is to study the operation properties of CRS and further study the corresponding algebraic properties. Throughout the research, some new conclusions and achievements presented in the paper may enrich the theory of CRS.

The rest of this paper is organized as follows. In Section 2, we briefly review some basic concepts of Pawlak rough sets and covering rough sets. In Section 3, we discuss the properties of the first type of CRS. Especially, we study the operation properties of the first type of covering rough sets with respect to minimally unary covering. In order to compute the intersection and union of CRS, two algorithms are constructed and an example is employed to illustrate the effectiveness of these algorithms. In Section 4, we discuss the properties of the second type of CRS. More importantly, we study the operation properties of the second type of covering rough sets with respect to maximally unary covering. Meanwhile, some corresponding algorithms are explored. In Section 5, as an application of operation properties of CRS, some meaningful algebraic properties of CRS are further studied. Finally, Section 6 concludes this study.

2. Preliminaries

The following recalls some necessary concepts and preliminaries of Pawlak rough sets and covering, which are required in the sequel of our work. More details can be seen in references [30, 38, 74].

2.1. Pawlak rough sets

(U, R) is referred to as an approximation space, where U is a non-empty finite set(also called the universe of discourse). denote $[x]_R = \{y|(x, y) \in R\}$, $U/R = \{[x]_R|x \in U\}$, then $[x]_R$ is called the equivalence class of x and the quotient set U/R is called the equivalence class set of U.

Definition 2.1.1 Let (U, R) be an approximation space. For each $X \subseteq U$,

$$\underline{R}(X) = \{x \in U \mid [x]_R \subseteq X\}, \quad \overline{R}(X) = \{x \in U \mid [x]_R \cap X \neq \emptyset\}$$

are called Pawlak lower and upper approximations of X with respect to the equivalence relation R, respectively.

Proposition 2.1.1 Let $\emptyset$ be the empty set and $\sim X$ the complement of X in U. Pawlak rough sets have the following properties.

(1L) $\underline{R}(U) = U$;	(1H) $\overline{R}(U) = U$;
(2L) $\underline{R}(\emptyset) = \emptyset$;	(2H) $\overline{R}(\emptyset) = \emptyset$;
(3L) $\underline{R}(X) \subseteq X$;	(3H) $X \subseteq \underline{R}(X)$;
(4L) $\underline{R}(\underline{R}(X)) = \underline{R}(X)$;	(4H) $\overline{R}(\overline{R}(X)) = \overline{R}(X)$;
(5L) $\underline{R}(X \cap Y) = \underline{R}(X) \cap \underline{R}(Y)$;	(5H) $\overline{R}(X \cup Y) = \overline{R}(X) \cup \overline{R}(Y)$;
(6L) $X \subseteq Y \Rightarrow \underline{R}(X) \subseteq \underline{R}(Y)$;	(6H) $X \subseteq Y \Rightarrow \overline{R}(X) \subseteq \overline{R}(Y)$;
(7L) $\underline{R}(\sim X) =\sim \overline{R}(X)$;	(7H) $\overline{R}(\sim X) =\sim \underline{R}(X)$;
(8L) $\forall K \in U/R, \underline{R}(K) = K$;	(8H) $\forall K \in U/R, \overline{R}(K) = K$.

2.2. Covering rough sets

In this subsection, we list some basic concepts about covering to be used in this paper.

Definition 2.2.1 Let U be the universe and $\mathcal{C}$ a family of nonempty subsets of U. If $\cup\mathcal{C} = U$, then $\mathcal{C}$ is called a covering of U. The ordered pair $(U, \mathcal{C})$ is called a covering approximation space.

Definition 2.2.2 Suppose $\mathcal{C}$ is a covering of U. A neighborhood system $\mathcal{C}_x$ of x is defined by:

$$\mathcal{C}_x = \{K \in \mathcal{C}|x \in K\}.$$

Definition 2.2.3 Let $\mathcal{C}$ be a covering of U and $x \in U$, the following two neighborhoods of x induced from the neighborhood system $\mathcal{C}_x$ are defined by:

$$n_0(\mathcal{C}_x) = \cap\{K|K \in \mathcal{C}_x\}, \quad n_1(\mathcal{C}_x) = \cup\{K|K \in \mathcal{C}_x\}.$$

The minimum neighborhood $n_0(\mathcal{C}_x)$ and the maximum neighborhood $n_1(\mathcal{C}_x)$ have been proposed and investigated by many researchers [33, 44, 45, 69, 70]. $n_0(\mathcal{C}_x)$ is called the neighborhood of x by Zhu [66, 67] and $n_1(\mathcal{C}_x)$ is called the indiscernibility neighborhood of x by Pomykala [44, 45].

Definition 2.2.4 Suppose $\mathcal{C}_x$ is the neighborhood system of x induced by a covering $\mathcal{C}$. The minimal description and maximal description of x are defined, respectively, by:

$$md(x) = \{K \in \mathcal{C}_x | (\forall S \in \mathcal{C}_x)(S \subseteq K \Rightarrow K = S)\},$$
$$MD(x) = \{K \in \mathcal{C}_x | (\forall S \in \mathcal{C}_x)(K \subseteq S \Rightarrow K = S)\}.$$

The minimal description $md(x)$ and the maximal description $Md(x)$ of x have been proposed and studied by many authors[2,44,74].

Definition 2.2.5 Let $\mathcal{C}$ be a covering of U. $\mathcal{C}$ is called minimally unary covering, if $\forall x \in U$, $|md(x)| = 1$. Meanwhile, the ordered pair $(U, \mathcal{C})$ is called a minimally unary covering approximation space.

In addition, if $\mathcal{C}$ is a minimally unary covering, then it is clear that $md(x) = \{n_0(\mathcal{C}_x)\}$, and for each $K \in \mathcal{C}_x$, we have that $n_0(\mathcal{C}_x) \subseteq K$. Moreover, we denote $\mathcal{C}^{min} = \{n_0(\mathcal{C}_x) | x \in U\}$.

In the following, we will employ an example to illustrate the Definition 2.2.5.

Example 2.2.1 A covering approximation space about hobby is given in Table 1. The universe $U = \{x_1, x_2, \cdots, x_6\}$ stands for six person. “Yes” means that the person likes the hobby. “No” means that the person does not like the hobby.

Table 1. A covering approximation space about hobby

U	Music	Sports	Drawing	Reading
x_1	Yes	No	No	No
x_2	Yes	Yes	Yes	No
x_3	No	Yes	No	No
x_4	No	No	No	Yes
x_5	Yes	Yes	Yes	No
x_6	No	No	No	Yes

Denote $K_M = \{x_1, x_2, x_5\}$, $K_S = \{x_2, x_3, x_5\}$, $K_D = \{x_2, x_5\}$, $K_R = \{x_4, x_6\}$. Clearly, $\mathcal{C} = \{K_M, K_S, K_D, K_R\}$ is a minimally unary covering of U and $(U, \mathcal{C})$ is a minimally unary covering approximation space. Moreover, $n_0(\mathcal{C}_{x_1}) = \{x_1, x_2, x_5\}$, $n_0(\mathcal{C}_{x_2}) = n_0(\mathcal{C}_{x_5}) = \{x_2, x_5\}$, $n_0(\mathcal{C}_{x_3}) = \{x_2, x_3, x_5\}$, $n_0(\mathcal{C}_{x_4}) = n_0(\mathcal{C}_{x_6}) = \{x_4, x_6\}$. Therefore, $\mathcal{C}^{min} = \{\{x_1, x_2, x_5\}, \{x_2, x_3, x_5\}, \{x_2, x_5\}, \{x_4, x_6\}\}$.

Definition 2.2.6 Let $\mathcal{C}$ be a covering of U. $\mathcal{C}$ is called maximally unary covering, if $\forall x \in U$, $|Md(x)| = 1$. Meanwhile, the ordered pair $(U, \mathcal{C})$ is called a maximally unary covering approximation space.

In addition, if $\mathcal{C}$ is a maximally unary covering, then it is clear that $Md(x) = \{n_1(\mathcal{C}_x)\}$, and for each $K \in \mathcal{C}_x$, we have that $K \subseteq n_1(\mathcal{C}_x)$. Moreover, we denote $\mathcal{C}^{max} = \{n_1(\mathcal{C}_x) | x \in U\}$.

Similarly, an example is presented to illustrate the Definition 2.2.6.

Example 2.2.2 The universe $U = \{x_1, x_2, \cdots, x_6\}$ stands for six students. A covering of U about course is given in Table 2. "Yes" means that the student has studied the course. "No" means that the student still does not study the course.

Table 2. A covering about course

U	History	Physics	Chemistry	Biology	Geography
x_1	Yes	No	No	No	No
x_2	No	Yes	Yes	No	No
x_3	No	No	Yes	Yes	No
x_4	No	No	Yes	Yes	Yes
x_5	Yes	No	No	No	No
x_6	No	No	Yes	No	Yes

Denote $K_H = \{x_1, x_5\}$, $K_P = \{x_2\}$, $K_C = \{x_2, x_3, x_4, x_6\}$, $K_B = \{x_3, x_4\}$, $K_G = \{x_4, x_6\}$. Clearly, $\mathcal{C} = \{K_H, K_P, K_C, K_B, K_G\}$ is a maximally unary covering of U and $(U, \mathcal{C})$ is a maximally unary covering approximation space. Moreover, $n_1(\mathcal{C}_{x_1}) = n_1(\mathcal{C}_{x_5}) = \{x_1, x_5\}, n_1(\mathcal{C}_{x_2}) = n_1(\mathcal{C}_{x_3}) = n_1(\mathcal{C}_{x_4}) = n_1(\mathcal{C}_{x_6}) = \{x_2, x_3, x_4, x_6\}$. Therefore, we have that $\mathcal{C}^{max} = \{\{x_1, x_5\}, \{x_2, x_3, x_4, x_6\}\}$.

Definition 2.2.7 Let $(U, \mathcal{C})$ be a covering approximation space. For each $X \subseteq U$,

$$\underline{C}(X) = \cup\{K \in \mathcal{C} | K \subseteq X\}, \quad \overline{C}(X) = \sim \underline{C}(\sim X).$$

are respectively called the first type of lower and upper covering approximations of X.

The ordered pair $(\underline{C}(X), \overline{C}(X))$ is called the first type of covering rough set of X. Clearly, $\mathbb{C}^F = \{(\underline{C}(X), \overline{C}(X)) \mid X \subseteq U\}$ is a set of all the first type of covering rough sets.

Meanwhile

$$\overline{C}(X) = \cup\{K \in \mathcal{C} | K \cap X \neq \emptyset\}, \quad \underline{C}(X) = \sim \overline{C}(\sim X).$$

are respectively called the second type of lower and upper covering approximations of X.

The ordered pair $(\underline{C}(X), \overline{C}(X))$ is called the second type of covering rough set of X. Therefore, $\mathbb{C}^S = \{(\underline{C}(X), \overline{C}(X) | X \subseteq U\}$ is a set of all the second type of covering rough sets.

The two pairs of covering rough sets listed in Definition 2.2.7 can go back to the papers by Pomykala [44,45]. But the explicit definition in terms of dual pairs has been given by Yao [64].

3. First type of covering rough sets

3.1. Properties of first type of covering approximation operators

In this subsection, we will study some basic properties of the first type of covering approximation operators in a covering approximation space.

According to the properties of Pawlak rough sets listed in Proposition 2.1.1, we have the following results.

Proposition 3.1.1 [47] Let $(U, \mathcal{C})$ be a covering approximation space and $X, Y \subseteq U$. Then, the following properties hold.

(1) $\underline{C}(U) = \overline{C}(U) = U$, $\underline{C}(\emptyset) = \overline{C}(\emptyset) = \emptyset$;

(2) $\underline{C}(X) \subseteq X \subseteq \overline{C}(X)$;

(3) $X \subseteq Y \Rightarrow \underline{C}(X) \subseteq \underline{C}(Y), \overline{C}(X) \subseteq \overline{C}(Y)$;

(4) $\underline{C}(\underline{C}(X) = \underline{C}(X), \overline{C}(\overline{C}(X) = \overline{C}(X)$;

(5) $\underline{C}(\sim X) =\sim \overline{C}(X), \overline{C}(\sim X) =\sim \underline{C}(X)$.

Remark 3.1.1 The properties (5L) and (5H) listed in Proposition 2.1.1 do not hold for the first type of covering rough sets. A counterexample is given as follows.

Example 3.1.1 Let $U = \{x_1, x_2, \cdots, x_6\}$, $K_1 = \{x_1, x_2, x_3\}$, $K_2 = \{x_1, x_4, x_5\}$, $K_3 = \{x_3, x_5, x_6\}$. Clearly, $\mathcal{C} = \{K_1, K_2, K_3\}$ is a covering of U. For $X_1 = \{x_1, x_3, x_5, x_6\}$, $Y_1 = \{x_1, x_2, x_3, x_4, x_5\}$, we have that $\underline{C}(X_1 \cap Y_1) = \emptyset$. But $\underline{C}(X_1) \cap \underline{C}(Y_1) = \{x_3, x_5\}$. Hence $\underline{C}(X_1 \cap Y_1) \neq \underline{C}(X_1) \cap \underline{C}(Y_1)$. For $X_2 = \{x_6\}$, $Y_2 = \{x_2, x_4\}$, we have that $\overline{C}(X_2 \cup Y_2) = U$. But $\overline{C}(X_2) \cup \overline{C}(Y_2) = \{x_1, x_2, x_4, x_6\}$. Hence $\overline{C}(X_2 \cup Y_2) \neq \overline{C}(X_2) \cup \overline{C}(Y_2)$.
In the following, we give a condition under which the properties (5L) and (5H) hold for the first type of covering rough sets.

Proposition 3.1.2 Let $(U, \mathcal{C})$ be a minimally unary covering approximation space and $X, Y \subseteq U$. Then, the following properties hold.

(1)$\underline{C}(X \cap Y) = \underline{C}(X) \cap \underline{C}(Y)$

(2)$\overline{C}(X \cup Y) = \overline{C}(X) \cup \overline{C}(Y)$

Proof:
(1) ($\Rightarrow$) It is evident by Proposition 3.1.1.

($\Leftarrow$) For each $x \in \underline{C}(X) \cap \underline{C}(Y)$, we have $x \in \underline{C}(X)$ and $x \in \underline{C}(Y)$. It can be found that $n_0(\mathcal{C}_x) \subseteq X$ and $n_0(\mathcal{C}_x) \subseteq Y$. Therefore, $n_0(\mathcal{C}_x) \subseteq X \cap Y$. By Definition 2.2.7, we have that $x \in \underline{C}(X \cap Y)$, i.e., $\underline{C}(X) \cap \underline{C}(Y) \subseteq \underline{C}(X \cap Y)$.

(2) The property can be proved similarly to (1). □

By Definition 2.2.7, the following result is obvious.

Proposition 3.1.3 Let $(U, \mathcal{C})$ be a minimally unary covering approximation space and $n_0(\mathcal{C}_x) \in \mathcal{C}^{min}$. Then, $n_0(\mathcal{C}_x) = \underline{C}(n_0(\mathcal{C}_x))$.

Proposition 3.1.4 Let $(U, \mathcal{C})$ be a minimally unary covering approximation space and $X, Y \subseteq U$. Then, the following properties hold.

(1)$\underline{C}(\underline{C}(X) \cap \underline{C}(Y)) = \underline{C}(X) \cap \underline{C}(Y)$; (2)$\underline{C}(\underline{C}(X) \cup \underline{C}(Y)) = \underline{C}(X) \cup \underline{C}(Y)$

(3)$\overline{C}(\overline{C}(X) \cap \overline{C}(Y)) = \overline{C}(X) \cap \overline{C}(Y)$; (4)$\overline{C}(\overline{C}(X) \cup \overline{C}(Y)) = \overline{C}(X) \cup \overline{C}(Y)$

Proof:
(1) ($\Rightarrow$) It is evident by Proposition 3.1.1.

($\Leftarrow$) For each $x \in \underline{C}(X)$, we have that $x \in n_0(\mathcal{C}_x) \subseteq X$. By Proposition 3.1.3, it follows that $x \in n_0(\mathcal{C}_x) = \underline{C}(n_0(\mathcal{C}_x)) \subseteq \underline{C}(X)$. Similarly, for each $x \in \underline{C}(Y)$, we have that $x \in n_0(\mathcal{C}_x) \subseteq \underline{C}(Y)$. Hence, for each $x \in \underline{C}(X) \cap \underline{C}(Y)$, it follows that $x \in n_0(\mathcal{C}_x) \subseteq \underline{C}(X) \cap \underline{C}(Y)$. By Definition 2.2.7, it can be obtained that $x \in \underline{C}(\underline{C}(X) \cap \underline{C}(Y))$. That is to say that $\underline{C}(X) \cap \underline{C}(Y) \subseteq \underline{C}(\underline{C}(X) \cap \underline{C}(Y))$.

(2) The property can be proved similarly to (1).

(3) This item can be proved by Definition 2.2.7 and item (1).

(4) This item can be proved by Definition 2.2.7 and item (2). □

Proposition 3.1.5 Let $(U, \mathcal{C})$ be a minimally unary covering approximation space, $x \in U$ and $X \subseteq U$. If $n_0(\mathcal{C}_x) = \{x\}$ and $x \in \overline{C}(X)$, then $x \in \underline{C}(X)$.

Proof:
Since $x \in \overline{C}(X)$, we have that $x \in X$ by Definition 2.2.7. That is to say that $\{x\} = n_0(\mathcal{C}_x) \subseteq X$. Then, it follows that $x \in \underline{C}(X)$. □

3.2. Operation properties of first type of covering rough sets

In this section, we will research the operation properties of the first type of covering rough sets. We first propose the concept of complement operation of covering rough sets.

Definition 3.2.1 Let $(U, \mathcal{C})$ be a covering approximation space. For each $(\underline{C}(X), \overline{C}(X)) \in \mathbb{C}^F$, the complement of it is defined as:

$$\sim (\underline{C}(X), \overline{C}(X)) = (\sim \overline{C}(X), \sim \underline{C}(X))$$

Remark 3.2.1 According to Proposition 3.1.1, we have $\sim (\underline{C}(X), \overline{C}(X)) = (\sim \overline{C}(X), \sim \underline{C}(X)) = (\underline{C}(\sim X), \overline{C}(\sim X))$. In other words, the complement of CRS of X is the CRS of $\sim X$.

Definition 3.2.2 Let $(U, \mathcal{C})$ be a covering approximation space. For $\forall (\underline{C}(X), \overline{C}(X)), (\underline{C}(Y), \overline{C}(Y)) \in \mathbb{C}^F$, the intersection and union of them are defined as:

(1) $(\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(Y), \overline{C}(Y)) = (\underline{C}(X) \cap \underline{C}(Y), \overline{C}(X) \cap \overline{C}(Y))$.

(2) $(\underline{C}(X), \overline{C}(X)) \cup (\underline{C}(Y), \overline{C}(Y)) = (\underline{C}(X) \cup \underline{C}(Y), \overline{C}(X) \cup \overline{C}(Y))$.

Is $\mathbb{C}^F$ closed under set intersection and union defined above? we will present an example to answer the question.

Example 3.2.1 Let $U = \{x_1, x_2, \cdots, x_6\}$, $K_1 = \{x_1, x_2, x_3\}$, $K_2 = \{x_3, x_5\}$, $K_3 = \{x_4, x_5, x_6\}$. Clearly, $\mathcal{C} = \{K_1, K_2, K_3\}$ is a covering of U. For $X = \{x_1, x_2, x_3\}$, $Y = \{x_3, x_4, x_5\}$, we have that $\underline{C}(X) \cap \underline{C}(Y) = \{x_3\}$, $\overline{C}(X) \cap \overline{C}(Y) = \{x_1, x_2, x_3\}$. It is obvious that there doesn't exist $V \subseteq U$ such that $(\underline{C}(V), \overline{C}(V)) = (\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(Y), \overline{C}(Y)) = (\underline{C}(X) \cap \underline{C}(Y), \overline{C}(X) \cap \overline{C}(Y))$. Similarly, for $X^{'} = \{x_1, x_2, x_6\}$, $Y^{'} = \{x_4, x_5, x_6\}$. It is clear that there doesn't exist $W \subseteq U$ such that $(\underline{C}(W), \overline{C}(W)) = (\underline{C}(X^{'}), \overline{C}(X^{'})) \cup (\underline{C}(Y^{'}), \overline{C}(Y^{'})) = (\underline{C}(X^{'}) \cup \underline{C}(Y^{'}), \overline{C}(X^{'}) \cup \overline{C}(Y^{'}))$.

Example 3.2.1 shows that $\mathbb{C}^F$ is not closed under set intersection and union. In order to solve the question listed above, we will raise the following two questions:

(Q1) Does there exist a condition under which $\mathbb{C}^F$ is closed under set intersection and union;

(Q2) If there exists the condition presented by (Q1). Then for $\forall(\underline{C}(X), \overline{C}(X)), (\underline{C}(Y), \overline{C}(Y)) \in \mathbb{C}^F$, can we compute two subsets $V, W \subseteq U$ such that $(\underline{C}(V), \overline{C}(V)) = (\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(Y), \overline{C}(Y))$ and $(\underline{C}(W), \overline{C}(W)) = (\underline{C}(X), \overline{C}(X)) \cup (\underline{C}(Y), \overline{C}(Y))$?

In what follows, we will devote to solving the questions (Q1) and (Q2).
Let $(U, \mathcal{C})$ be a minimally unary covering approximation space and for $\forall x, y \in U$, we have $K_x^{min} = K_y^{min}$ or $K_x^{min} \cap K_y^{min} = \emptyset$. Denote

$A = A_2/A_1$, where $A_1 = \underline{C}(X) \cap \underline{C}(Y), A_2 = \overline{C}(X) \cap \overline{C}(Y)$. $\mathcal{A} = \{n_0(\mathcal{C}_x) | x \in A, n_0(\mathcal{C}_x) \cap A_1 = \emptyset\}$. $\mathcal{A}' = \{n_0(\mathcal{C}_{x_i}) | x_i \in A, i = 1, 2, \cdots, m\}$, where $\mathcal{A}'$ satisfies the following three conditions.

(1) For each $n_0(\mathcal{C}_{x_i}) \in \mathcal{A}'$, where $i = 1, 2, \cdots, m$, we have $n_0(\mathcal{C}_{x_i}) \in \mathcal{A}$;

(2) For each $n_0(\mathcal{C}_{x_i}), n_0(\mathcal{C}_{x_j}) \in \mathcal{A}'$, where $i \neq j; i, j = 1, 2, \cdots, m$, we have that $n_0(\mathcal{C}_{x_i}) \cap n_0(\mathcal{C}_{x_j}) = \emptyset$;

(3) For each $n_0(\mathcal{C}_x) \in \mathcal{A}$, there exists $n_0(\mathcal{C}_{x_i}) \in \mathcal{A}'$ such that $n_0(\mathcal{C}_{x_i}) \subseteq K_x^{min}$.

Furthermore, denote $P = \{x_i | n_0(\mathcal{C}_{x_i}) \in \mathcal{A}', i = 1, 2, \cdots, m\}$ and $V = A_1 \cup P$.

On the other hand, denote $B = B_2/B_1$, where $B_1 = \underline{C}(X) \cup \underline{C}(Y), B_2 = \overline{C}(X) \cup \overline{C}(Y)$. $\mathcal{B} = \{n_0(\mathcal{C}_x) | x \in B, n_0(\mathcal{C}_x) \cap B_1 = \emptyset\}$. $\mathcal{B}' = \{n_0(\mathcal{C}_{x_j}) | x_j \in B, j = 1, 2, \cdots, n\}$, where $\mathcal{B}'$ satisfies the following three conditions.

(1) For each $n_0(\mathcal{C}_{x_j}) \in \mathcal{B}'$, where $j = 1, 2, \cdots, n$, we have $n_0(\mathcal{C}_{x_j}) \in \mathcal{B}$;

(2) For each $n_0(\mathcal{C}_{x_i}), n_0(\mathcal{C}_{x_j}) \in \mathcal{B}'$, where $i \neq j; i, j = 1, 2, \cdots, n$, we have that $n_0(\mathcal{C}_{x_i}) \cap n_0(\mathcal{C}_{x_j}) = \emptyset$;

(3) For each $n_0(\mathcal{C}_x) \in \mathcal{B}$, there exists $K_{x_j}^{min} \in \mathcal{B}'$ such that $n_0(\mathcal{C}_{x_j}) \subseteq n_0(\mathcal{C}_x)$.

Furthermore, denote $Q = \{x_j | n_0(\mathcal{C}_x) \in \mathcal{B}', j = 1, 2, \cdots, n\}$ and $W = B_1 \cup Q$.
According to the approach presented above, we can compute two subsets $V, W \subseteq U$ such that the following properties hold.

Proposition 3.2.1 Let $(U, \mathcal{C})$ be a minimally unary covering approximation space and for $\forall x, y \in U$, we have $K_x^{min} = K_y^{min}$ or $K_x^{min} \cap K_y^{min} = \emptyset$. For V, W defined above, then

(1) $\underline{C}(V) = \underline{C}(X) \cap \underline{C}(Y)$; (2) $\overline{C}(V) = \overline{C}(X) \cap \overline{C}(Y)$;

(3) $\underline{C}(W) = \underline{C}(X) \cup \underline{C}(Y)$; (4) $\overline{C}(W) = \overline{C}(X) \cup \overline{C}(Y)$.

Proof:
(1) ($\Rightarrow$) For each $x \in \underline{C}(V)$, we have that $n_0(\mathcal{C}_x) \subseteq V$. By the construction of V and Proposition 3.1.5, it follows that $x \in n_0(\mathcal{C}_x) \subseteq \underline{C}(X) \cap \underline{C}(Y)$. Hence, $\underline{C}(V) \subseteq \underline{C}(X) \cap \underline{C}(Y)$.

($\Leftarrow$) It is clear that $\underline{C}(X) \cap \underline{C}(Y) \subseteq V$. By Proposition 3.1.1 and Proposition 3.1.4, we have that $\underline{C}(X) \cap \underline{C}(Y) = \underline{C}(\underline{C}(X) \cap \underline{C}(Y)) \subseteq \underline{C}(V)$. i.e., $\underline{C}(X) \cap \underline{C}(Y) \subseteq \underline{C}(V)$.

(2) ($\Rightarrow$) It is obvious that $V \subseteq \overline{C}(X) \cap \overline{C}(Y)$. By Proposition 3.1.1 and Proposition 3.1.4, we have that $\overline{C}(V) \subseteq \overline{C}(\overline{C}(X) \cap \overline{C}(Y)) = \overline{C}(X) \cap \overline{C}(Y)$. i.e., $\overline{C}(V) \subseteq \overline{C}(X) \cap \overline{C}(Y)$.

($\Leftarrow$) It is evident by Definition 2.2.7 and the construction of V.

(3) This item can be proved similarly to (1).

(4) The property can be proved similarly to (2). □

In order to better understand Proposition 3.2.1 and the relationship between subset V and subset W, the following remark is shown:

Remark 3.2.2 (1) If $\mathcal{C}$ is a minimally unary covering, there must exist two subsets $V, W \subseteq U$ such that $(\underline{C}(V), \overline{C}(V)) = (\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(Y), \overline{C}(Y)) = (\underline{C}(X) \cap \underline{C}(Y), \overline{C}(X) \cap \overline{C}(Y))$ and $(\underline{C}(W), \overline{C}(W)) = (\underline{C}(X), \overline{C}(X)) \cup (\underline{C}(Y), \overline{C}(Y)) = (\underline{C}(X) \cup \underline{C}(Y), \overline{C}(X) \cup \overline{C}(Y))$.

(2) According to the construction of V, W, we have $\underline{C}(X) \cap \underline{C}(Y) \subseteq V \subseteq \overline{C}(X) \cap \overline{C}(Y)$ and $\underline{C}(X) \cup \underline{C}(Y) \subseteq W \subseteq \overline{C}(X) \cup \overline{C}(Y)$.

(3) For each subset V, we can find a subset W such that $V \subseteq W$.

To apply this approach to practical issues, we here present two algorithms for computing subsets V and W.

Algorithm 1: An algorithm for computing subset V

Input : A minimally unary covering approximation space $(U, \mathcal{C})$ and $X, Y \subseteq U$;
Output : Subset V.

begin
    Compute $\underline{C}(X) \cap \underline{C}(Y), \overline{C}(X) \cap \overline{C}(Y), (\overline{C}(X) \cap \overline{C}(Y))/(\underline{C}(X) \cap \underline{C}(Y))$;
    Compute $n_0(\mathcal{C}_{x_1}), n_0(\mathcal{C}_{x_2}), \cdots, n_0(\mathcal{C}_{x_s})$;
    $K \leftarrow \emptyset$;
    for $i = 1 : s; i <= s; i++$ **do**
        if $n_0(\mathcal{C}_{x_i}) \cap (\underline{C}(X) \cap \underline{C}(Y)) = \emptyset$ **then**
            $K \leftarrow K \cup \{x_i\}$;
        end
    end
    $n_0(\mathcal{C}_x) \leftarrow \emptyset; K^{min} \leftarrow \emptyset; T \leftarrow \emptyset$;
    if $|K| \neq 0$ **then**
        for $j = 1 : |K|; j <= |K|; j++$ **do**
            for *each* $x \in K$ **do**
                if $n_0(\mathcal{C}_x) \cap n_0(\mathcal{C}_{x_j}) \neq \emptyset$ **then**
                    $n_0(\mathcal{C}_x) \leftarrow n_0(\mathcal{C}_x) \cap n_0(\mathcal{C}_{x_j})$;
                    $K^{min} \leftarrow K^{min} \cup n_0(\mathcal{C}_{x_j})$;
                    $T \leftarrow T \cup \{x\}$;
                end
                $K \leftarrow K/K^{min}$;
            end
        end
    end
    Compute $(\underline{C}(X) \cap \underline{C}(Y)) \cap T$; // $V = (\underline{C}(X) \cap \underline{C}(Y)) \cap T$ by the construction of V;
end

In the following, an algorithm for computing subset W presented above are constructed.

Algorithm 2: An algorithm for computing subset W

Input : A minimally unary covering approximation space $(U, \mathcal{C})$ and $X, Y \subseteq U$;
Output : Subset W.

begin
  Compute $\underline{C}(X) \cup \underline{C}(Y), \overline{C}(X) \cup \overline{C}(Y), (\overline{C}(X) \cup \overline{C}(Y))/(\underline{C}(X) \cup \underline{C}(Y))$;
  Compute $n_0(\mathcal{C}_{x_1}), n_0(\mathcal{C}_{x_2}), \cdots, n_0(\mathcal{C}_{x_s})$;
  $K \leftarrow \emptyset$;
  for $i = 1 : s; i <= s; i++$ **do**
    if $n_0(\mathcal{C}_{x_i}) \cap (\underline{C}(X) \cup \underline{C}(Y)) = \emptyset$ **then**
      $K \leftarrow K \cup \{x_i\}$;
    end
  end
  $n_0(\mathcal{C}_x) \leftarrow \emptyset; K^{min} \leftarrow \emptyset; T \leftarrow \emptyset$;
  if $|K| \neq 0$ **then**
    for $j = 1 : |K|; j <= |K|; j++$ **do**
      for *each* $x \in K$ **do**
        if $n_0(\mathcal{C}_x) \cap n_0(\mathcal{C}_{x_j}) \neq \emptyset$ **then**
          $n_0(\mathcal{C}_x) \leftarrow n_0(\mathcal{C}_x) \cap n_0(\mathcal{C}_{x_j})$;
          $K^{min} \leftarrow K^{min} \cup n_0(\mathcal{C}_{x_j})$;
          $T \leftarrow T \cup \{x\}$;
        end
        $K \leftarrow K/K^{min}$;
      end
    end
  end
  Compute $(\underline{C}(X) \cup \underline{C}(Y)) \cup T$; // $W = (\underline{C}(X) \cup \underline{C}(Y)) \cup T$ by the construction of W;
end

Then, we will employ an example to show the effectiveness of the two algorithms.

Example 3.2.2 Let $U = \{x_1, x_2, \cdots, x_{10}\}, \mathcal{C} = \{\{x_1, x_2, x_3, x_4\}, \{x_4\}, \{x_4, x_5, x_6, x_7\}, \{x_8, x_9, x_{10}\}\}$. For $X = \{x_1, x_4\}, Y = \{x_4, x_5\}$, let $V = \{x_1, x_4, x_5\}, W = \{x_1, x_4, x_5\}$, then we have that

$$(\underline{C}(V), \overline{C}(V)) = (\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(Y), \overline{C}(Y)) \quad \cdots (1)$$
$$(\underline{C}(W), \overline{C}(W)) = (\underline{C}(X), \overline{C}(X)) \cup (\underline{C}(Y), \overline{C}(Y)) \cdots (2)$$

It is clear that the selections of V, W satisfying equations (1) and (2) are not unique. For $X = \{x_1, x_4\}, Y = \{x_4, x_5\}$, all the selections of V, W obtained probably by Algorithms 1 and 2 are shown in Table 3.

Table 3 All the selections of V, W

X, Y	V	W
$\{x_1, x_4\}$	$\{x_1, x_4, x_5\}$	$\{x_1, x_4, x_5\}$
$\{x_4, x_5\}$	$\{x_2, x_4, x_5\}$	$\{x_2, x_4, x_5\}$
	$\{x_3, x_4, x_5\}$	$\{x_3, x_4, x_5\}$
	$\{x_1, x_4, x_6\}$	$\{x_1, x_4, x_6\}$
	$\{x_2, x_4, x_6\}$	$\{x_2, x_4, x_6\}$
	$\{x_3, x_4, x_6\}$	$\{x_3, x_4, x_6\}$
	$\{x_1, x_4, x_7\}$	$\{x_1, x_4, x_7\}$
	$\{x_2, x_4, x_7\}$	$\{x_2, x_4, x_7\}$
	$\{x_3, x_4, x_7\}$	$\{x_3, x_4, x_7\}$

Furthermore, for $Z = \{x_5, x_8\}$. Let $C = \{x_5\}, D = \{x_1, x_4, x_5, x_8\}$, then we have that

$$(\underline{C}(C), \overline{C}(C)) = ((\underline{C}(X), \overline{C}(X)) \cup (\underline{C}(Y), \overline{C}(Y))) \cap (\underline{C}(Z), \overline{C}(Z)) \cdots (3)$$
$$(\underline{C}(D), \overline{C}(D)) = ((\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(Y), \overline{C}(Y))) \cup (\underline{C}(Z), \overline{C}(Z)) \cdots (4)$$

Similarly, the selections of C, D which satisfy equations (3) and (4) are also not unique. For $X = \{x_1, x_4\}, Y = \{x_4, x_5\}, Z = \{x_1, x_5, x_6\}$, all the selections of C, D obtained probably by Algorithms 1 and 2 are shown in Table 4.

Table 4 All the selections of C, D

X, Y, Z	C	D	
$\{x_1, x_4\}$	$\{x_5\}$	$\{x_1, x_4, x_5, x_8\}$	$\{x_2, x_4, x_5, x_8\}$
$\{x_4, x_5\}$	$\{x_6\}$	$\{x_3, x_4, x_5, x_8\}$	$\{x_1, x_4, x_6, x_8\}$
$\{x_5, x_8\}$	$\{x_7\}$	$\{x_2, x_4, x_6, x_8\}$	$\{x_3, x_4, x_6, x_8\}$
		$\{x_1, x_4, x_7, x_8\}$	$\{x_2, x_4, x_7, x_8\}$
		$\{x_3, x_4, x_7, x_8\}$	$\{x_1, x_4, x_5, x_9\}$
		$\{x_2, x_4, x_5, x_9\}$	$\{x_3, x_4, x_5, x_9\}$
		$\{x_1, x_4, x_6, x_9\}$	$\{x_2, x_4, x_6, x_9\}$
		$\{x_3, x_4, x_6, x_9\}$	$\{x_1, x_4, x_7, x_9\}$
		$\{x_2, x_4, x_7, x_9\}$	$\{x_3, x_4, x_7, x_9\}$
		$\{x_1, x_4, x_5, x_{10}\}$	$\{x_2, x_4, x_5, x_{10}\}$
		$\{x_3, x_4, x_5, x_{10}\}$	$\{x_1, x_4, x_6, x_{10}\}$
		$\{x_2, x_4, x_6, x_{10}\}$	$\{x_3, x_4, x_6, x_{10}\}$
		$\{x_1, x_4, x_7, x_{10}\}$	$\{x_2, x_4, x_7, x_{10}\}$
		$\{x_3, x_4, x_7, x_{10}\}$	

4. Second type of covering rough sets

4.1. Properties of second type of covering approximation operators

In this subsection, we will study the properties of the second type of covering approximation operators in a covering approximation space.

Proposition 4.1.1 Let $(U, \mathcal{C})$ be a covering approximation space and $X \subseteq U$. Then we have $\underline{C}(X) = \cap\{\sim K | K \in \mathcal{C}, K \nsubseteq X\}$.

Proof:
$\underline{C}(X) =\sim \overline{C}(\sim X) =\sim \{\cup\{K \in \mathcal{C} | K \cap (\sim X) \neq \emptyset\}\} =\sim \{\cup\{K \in \mathcal{C} | K \nsubseteq X\}\} = \cap\{\sim \{K \in \mathcal{C} | K \nsubseteq X\}\} = \cap\{\sim K | K \in \mathcal{C}, K \nsubseteq X\}$. □

To illustrate the above proposition, an example is presented as follows.

Example 4.1.1 Let $U = \{x_1, x_2, \cdots, x_6\}, \mathcal{C} = \{\{x_1, x_2\}, \{x_2, x_3, x_4\}, \{x_4, x_5\}, \{x_5, x_6\}\}$. For $X = \{x_5, x_6\}$, according to Proposition 4.1, we have $\underline{C}(X) =\sim \{x_1, x_2\} \cap \sim \{x_2, x_3, x_4\} \cap \sim \{x_4, x_5\} = \{x_6\}$.
Similar to Proposition 3.1.1, we have the following results.

Proposition 4.1.2 Let $(U, \mathcal{C})$ be a covering approximation space and $X, Y \subseteq U$. Then, the following properties hold.

(1) $\underline{C}(U) = \overline{C}(U) = U, \underline{C}(\emptyset) = \overline{C}(\emptyset) = \emptyset$;
(2) $\underline{C}(X) \subseteq X \subseteq \overline{C}(X)$;
(3) $X \subseteq Y \Rightarrow \underline{C}(X) \subseteq \underline{C}(Y), \overline{C}(X) \subseteq \overline{C}(Y)$;
(4) $\underline{C}(X \cap Y) = \underline{C}(X) \cap \underline{C}(Y),\ \ \overline{C}(X \cup Y) = \overline{C}(X) \cup \overline{C}(Y)$;
(5) $\underline{C}(\sim X) =\sim \overline{C}(X), \overline{C}(\sim X) =\sim \underline{C}(X)$.

Remark 4.1.1 The properties (4L) and (4H) do not hold for the second type of covering rough sets. A counterexample is shown as follows.

Example 4.1.2 (Continued from Example 4.1) We have $\underline{C}(\underline{C}(X)) = \emptyset \neq \{x_6\} = \underline{C}(X), \overline{C}(\overline{C}(X)) = \{x_2, x_3, x_4, x_5, x_6\} \neq \{x_4, x_5, x_6\} = \overline{C}(X)$.
Before we explore the properties (4L) and (4H) for the second type of covering rough sets, two lemmas will be proposed. The proofs of the two lemmas straightforwardly follow by the notions involved and are thus omitted.

Lemma 4.1.1 Let $(U, \mathcal{C})$ be a maximally unary covering-based approximation space and $n_1(\mathcal{C}_x)$, $n_1(\mathcal{C}_y) \in \mathcal{C}^{max}$. Then we have that $n_1(\mathcal{C}_x) \cap n_1(\mathcal{C}_y) = \emptyset$ or $n_1(\mathcal{C}_x) = n_1(\mathcal{C}_y)$.

Lemma 4.1.2 Let $(U, \mathcal{C})$ be a maximally unary covering-based approximation space and $x \in U, X \subseteq U$. Then, the following properties hold.

(1) $\overline{C}(n_1(\mathcal{C}_x)) = n_1(\mathcal{C}_x)$;
(2) $\overline{C}(X) = \cup_{x \in X} n_1(\mathcal{C}_x)$.

Proposition 4.1.3 Let $(U, \mathcal{C})$ be a maximally unary covering approximation space and $X \subseteq U$. Then, the following properties hold.

(1) $\overline{C}(\overline{C}(X)) = \overline{C}(X)$;

(2) $\underline{C}(\underline{C}(X)) = \underline{C}(X)$.

Proof:
(1) By Lemma 4.1.2, we have $\overline{C}(X) = \cup_{x\in X} n_1(\mathcal{C}_x)$. According to Proposition 4.1.2 and Lemma 4.1.2, it follows that $\overline{C}(\overline{C}(X)) = \overline{C}(\cup_{x\in X} n_1(\mathcal{C}_x)) = \cup_{x\in X}\overline{C}(n_1(\mathcal{C}_x)) = \cup_{x\in X} n_1(\mathcal{C}_x) = \overline{C}(X)$. Hence, $\overline{C}(\overline{C}(X)) = \overline{C}(X)$.

(2) The property can be proved by item (1) and Proposition 4.1.2. □

Proposition 4.1.4 Let $(U, \mathcal{C})$ be a maximally unary covering approximation space and $X, Y \subseteq U$. Then, the following properties hold.

(1) $\overline{C}(\overline{C}(X) \cap \overline{C}(Y)) = \overline{C}(X) \cap \overline{C}(Y)$; (2) $\overline{C}(\overline{C}(X) \cup \overline{C}(Y)) = \overline{C}(X) \cup \overline{C}(Y)$;

(3) $\underline{C}(\underline{C}(X) \cap \underline{C}(Y)) = \underline{C}(X) \cap \underline{C}(Y)$; (4) $\underline{C}(\underline{C}(X) \cup \underline{C}(Y)) = \underline{C}(X) \cup \underline{C}(Y)$.

Proof:
(1) ($\Leftarrow$) It is easy to prove by Proposition 4.1.2.

($\Rightarrow$) By Propositions 4.1.2 and 4.1.3, we have $\overline{C}(\overline{C}(X) \cap \overline{C}(Y)) \subseteq \overline{C}(\overline{C}(X)) \cap \overline{C}(\overline{C}(Y)) = \overline{C}(X) \cap \overline{C}(Y)$.

(2) By Propositions 4.1.2 and 4.1.3, we have $\overline{C}(\overline{C}(X) \cup \overline{C}(Y)) = \overline{C}(\overline{C}(X)) \cup \overline{C}(\overline{C}(Y)) = \overline{C}(X) \cup \overline{C}(Y)$.

(3) This item can be proved similarly to (2).

(4) This item can be proved similarly to (1). □

Similar to Proposition 3.1.5, we have the following result.

Proposition 4.1.5 Let $(U, \mathcal{C})$ be a maximally unary covering approximation space, $x \in U$ and $X \subseteq U$. If $K_x^{max} = \{x\}$ and $x \in \overline{C}(X)$, then $x \in \underline{C}(X)$.

4.2. Operation properties of second type of CRS

In this section, we will research the operation properties of the second type of covering rough sets.

Clearly, according to Proposition 4.1.2, $\mathbb{C}^S$ is closed under set complement defined by Definition 3.2.1.

Is $\mathbb{C}^S$ closed under set intersection and union defined by Definition 3.2.2? we will present an example to answer the question.

Example 4.2.1 Let $U = \{x_1, x_2, \cdots, x_6\}$, $\mathcal{C} = \{\{x_1, x_2, x_3\}, \{x_3, x_5\}, \{x_4, x_5, x_6\}\}$ be a covering of U. For $X = \{x_1, x_2, x_3\}$, $Y = \{x_4, x_5\}$, we have that $\underline{C}(X) = \{x_1, x_2\}, \overline{C}(X) = \{x_1, x_2, x_3, x_5\}, \underline{C}(Y) = \emptyset, \overline{C}(Y) = \{x_3, x_4, x_5, x_6\}$. It is obvious that there doesn't exist $V \subseteq U$ such that $(\underline{C}(V), \overline{C}(V)) = (\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(Y), \overline{C}(Y)) = (\underline{C}(X) \cap \underline{C}(Y), \overline{C}(X) \cap \overline{C}(Y))$.

Meanwhile, for $X^{'} = \{x_1, x_2, x_3, x_6\}, Y^{'} = \{x_4, x_5, x_6\}$. It is clear that there doesn't exist $W \subseteq U$ such that $(\underline{C}(W), \overline{C}(W)) = (\underline{C}(X^{'}), \overline{C}(X^{'})) \cup (\underline{C}(Y^{'}), \overline{C}(Y^{'})) = (\underline{C}(X^{'}) \cup \underline{C}(Y^{'}), \overline{C}(X^{'}) \cup \overline{C}(Y^{'}))$.

Example 4.2.1 tells us that $\mathbb{C}^S$ is not closed under set intersection and union. Therefore, we can raise the following two questions:

(Q1) Does there exist a condition under which $\mathbb{C}^S$ is closed under set intersection and union;

(Q2) If there exists the condition shown by (Q1). Then for $\forall(\underline{C}(X), \overline{C}(X)), (\underline{C}(Y), \overline{C}(Y)) \in \mathbb{C}^S$, can we compute two subsets $V, W \subseteq U$ such that $(\underline{C}(V), \overline{C}(V)) = (\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(Y), \overline{C}(Y))$ and $(\underline{C}(W), \overline{C}(W)) = (\underline{C}(X), \overline{C}(X)) \cup (\underline{C}(Y), \overline{C}(Y))$?

In the following, we will devote to solving the questions (Q1) and (Q2).

Let $(U, \mathcal{C})$ be a maximally unary covering approximation space and $X, Y \subseteq U$. Denote

$A = A_2/A_1$, where $A_1 = \underline{C}(X) \cap \underline{C}(Y), A_2 = \overline{C}(X) \cap \overline{C}(Y)$.

$\mathcal{A} = \{n_1(\mathcal{C}_x) | x \in A, n_1(\mathcal{C}_x) \cap A_1 = \emptyset\}$.

$\mathcal{A}^{'} = \{n_1(\mathcal{C}_{x_i}) | x_i \in A, i = 1, 2, \cdots, m\}$, where $\mathcal{A}^{'}$ satisfies the following three conditions.

(1) For each $n_1(\mathcal{C}_x) \in \mathcal{A}^{'}$, where $i = 1, 2, \cdots, m$, we have $n_1(\mathcal{C}_x) \in \mathcal{A}$.

(2) For each $n_1(\mathcal{C}_{x_i}), n_1(\mathcal{C}_{x_j}) \in \mathcal{A}^{'}$, where $i \neq j; i, j = 1, 2, \cdots, m$, we have that $n_1(\mathcal{C}_{x_i}) \cap n_1(\mathcal{C}_{x_j}) = \emptyset$.

(3) For each $n_1(\mathcal{C}_x) \in \mathcal{A}$, there exists $n_1(\mathcal{C}_{x_i}) \in \mathcal{A}^{'}$ such that $n_1(\mathcal{C}_{x_i}) = n_1(\mathcal{C}_x)$.

Furthermore, denote $P = \{x_i | n_1(\mathcal{C}_{x_i}) \in \mathcal{A}^{'}, i = 1, 2, \cdots, m\}$ and $V = A_1 \cup P$.

On the other hand, denote

$B = B_2/B_1$, where $B_1 = \underline{C}(X) \cup \underline{C}(Y), B_2 = \overline{C}(X) \cup \overline{C}(Y)$.

$\mathcal{B} = \{n_1(\mathcal{C}_x) | x \in B, n_1(\mathcal{C}_x) \cap B_1 = \emptyset\}$.

$\mathcal{B}^{'} = \{n_1(\mathcal{C}_{x_j}) | x_j \in B, j = 1, 2, \cdots, n\}$, where $\mathcal{B}^{'}$ satisfies the following three conditions.

(1) For each $n_1(\mathcal{C}_{x_j}) \in \mathcal{B}^{'}$, where $j = 1, 2, \cdots, n$, we have $n_1(\mathcal{C}_{x_j}) \in \mathcal{B}$.

(2) For each $n_1(\mathcal{C}_{x_i}), n_1(\mathcal{C}_{x_j}) \in \mathcal{B}^{'}$, where $i \neq j; i, j = 1, 2, \cdots, n$, we have that $n_1(\mathcal{C}_{x_i}) \cap n_1(\mathcal{C}_{x_j}) = \emptyset$.

(3) For each $n_1(\mathcal{C}_x) \in \mathcal{B}$, there exists $n_1(\mathcal{C}_{x_j}) \in \mathcal{B}^{'}$ such that $n_1(\mathcal{C}_{x_j}) = n_1(\mathcal{C}_x)$.

Furthermore, denote $Q = \{x_j | n_1(\mathcal{C}_{x_j}) \in \mathcal{B}^{'}, j = 1, 2, \cdots, n\}$ and $W = B_1 \cup Q$.

According to the approach presented above, we can prove that $\mathbb{C}^S$ is closed under set intersection and union. Hence, similar to Proposition 3.2.1, we have the following proposition.

Proposition 4.2.1 Let $(U, \mathcal{C})$ be a maximally unary covering approximation space and $X, Y \subseteq U$. For V, W defined above, then the following properties hold.

(1) $\underline{C}(V) = \underline{C}(X) \cap \underline{C}(Y)$

(2) $\overline{C}(V) = \overline{C}(X) \cap \overline{C}(Y)$

(3) $\underline{C}(W) = \underline{C}(X) \cup \underline{C}(Y)$

(4) $\overline{C}(W) = \overline{C}(X) \cup \overline{C}(Y)$

Remark 4.2.1 (1) If $\mathcal{C}$ is a maximally unary covering, for $\forall(\underline{C}(X), \overline{C}(X))$, $(\underline{C}(Y), \overline{C}(Y)) \in \mathbb{C}^S$, we can find subsets $V, W \subseteq U$ such that $(\underline{C}(V), \overline{C}(V)) = (\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(Y), \overline{C}(Y)) = (\underline{C}(X) \cap \underline{C}(Y), \overline{C}(X) \cap \overline{C}(Y))$ and $(\underline{C}(W), \overline{C}(W)) = (\underline{C}(X), \overline{C}(X)) \cup (\underline{C}(Y), \overline{C}(Y)) = (\underline{C}(X) \cup \underline{C}(Y), \overline{C}(X) \cup \overline{C}(Y))$.

(2) According to the construction of V, W, we have $\underline{C}(X) \cap \underline{C}(Y) \subseteq V \subseteq \overline{C}(X) \cap \overline{C}(Y)$ and $\underline{C}(X) \cup \underline{C}(Y) \subseteq W \subseteq \overline{C}(X) \cup \overline{C}(Y)$.

(3) For each subset V, we can find a subset W such that $V \subseteq W$.

Similar to Algorithms 1 and 2, we can also construct corresponding algorithms to compute subsets V, W. Therefore, we won't repeat them here.

Example 4.2.2 Let $U = \{x_1, x_2, \cdots, x_{10}\}, \mathcal{C} = \{\{x_1\}, \{x_1, x_2, x_3\}, \{x_4, x_5\}, \{x_4, x_5, x_6, x_7\}, \{x_8\}, \{x_8, x_9, x_{10}\}\}$. For $X = \{x_1, x_4\}, Y = \{x_4, x_5\}$, we have that $\underline{C}(X) = \emptyset, \overline{C}(X) = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7\}$, $\underline{C}(Y) = \emptyset, \overline{C}(Y) = \{x_4, x_5, x_6, x_7\}$. Let $V = \{x_4\}, W = \{x_1, x_4\}$, then we have that

$$(\underline{C}(V), \overline{C}(V)) = (\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(Y), \overline{C}(Y)) \quad \cdots (5)$$
$$(\underline{C}(W), \overline{C}(W)) = (\underline{C}(X), \overline{C}(X)) \cup (\underline{C}(Y), \overline{C}(Y)) \cdots (6)$$

It is clear that the selections of V, W satisfying equations (5) and (6) are not unique. For $X = \{x_3, x_5\}, Y = \{x_2, x_3\}$, all the selections of V, W obtained probably by Proposition 4.2.1 are given in Table 5.

Table 5 All the selections of V, W

X, Y	V	W
$\{x_1, x_4\}$	$\{x_4\}$	$\{x_1, x_4\}$ $\{x_2, x_4\}$
$\{x_4, x_5\}$	$\{x_5\}$	$\{x_3, x_4\}$ $\{x_1, x_5\}$
	$\{x_6\}$	$\{x_2, x_5\}$ $\{x_3, x_5\}$
	$\{x_7\}$	$\{x_1, x_6\}$ $\{x_2, x_6\}$
		$\{x_3, x_6\}$ $\{x_1, x_7\}$
		$\{x_2, x_7\}$ $\{x_3, x_7\}$

Furthermore, for $Z = \{x_1, x_4, x_8\}$. Let $C = \{x_1, x_4\}, D = \{x_1, x_4, x_8\}$, then we have that

$$(\underline{C}(C), \overline{C}(C)) = ((\underline{C}(X), \overline{C}(X)) \cup (\underline{C}(Y), \overline{C}(Y))) \cap (\underline{C}(Z), \overline{C}(Z)) \cdots (7)$$
$$(\underline{C}(D), \overline{C}(D)) = ((\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(Y), \overline{C}(Y))) \cup (\underline{C}(Z), \overline{C}(Z)) \cdots (8)$$

Similarly, the selections of C, D which satisfy equations (7) and (8) are also not unique. For $X = \{x_1, x_4\}, Y = \{x_4, x_5\}, Z = \{x_1, x_4, x_8\}$, all the selections of C, D obtained probably by Proposition 4.2.1 are given in Table 6.

Table 6 All the selections of C, D

X, Y, Z	C	D
$\{x_1, x_4\}$	$\{x_1, x_4\}$	$\{x_1, x_4, x_8\}$ $\{x_2, x_4, x_8\}$
$\{x_4, x_5\}$	$\{x_2, x_4\}$	$\{x_3, x_4, x_8\}$ $\{x_1, x_5, x_8\}$
$\{x_1, x_4, x_8\}$	$\{x_3, x_4\}$	$\{x_2, x_5, x_8\}$ $\{x_3, x_5, x_8\}$
	$\{x_1, x_5\}$	$\{x_1, x_6, x_8\}$ $\{x_2, x_6, x_8\}$
	$\{x_2, x_5\}$	$\{x_3, x_6, x_8\}$ $\{x_1, x_7, x_8\}$
	$\{x_3, x_5\}$	$\{x_2, x_7, x_8\}$ $\{x_3, x_7, x_8\}$
	$\{x_1, x_6\}$	$\{x_1, x_4, x_9\}$ $\{x_2, x_4, x_9\}$
	$\{x_2, x_6\}$	$\{x_3, x_4, x_9\}$ $\{x_1, x_5, x_9\}$
	$\{x_3, x_6\}$	$\{x_2, x_5, x_9\}$ $\{x_3, x_5, x_9\}$
	$\{x_1, x_7\}$	$\{x_1, x_6, x_9\}$ $\{x_2, x_6, x_9\}$
	$\{x_2, x_7\}$	$\{x_3, x_6, x_9\}$ $\{x_1, x_7, x_9\}$
	$\{x_3, x_7\}$	$\{x_2, x_7, x_9\}$ $\{x_3, x_7, x_9\}$
		$\{x_1, x_4, x_{10}\}$ $\{x_2, x_4, x_{10}\}$
		$\{x_3, x_4, x_{10}\}$ $\{x_1, x_5, x_{10}\}$
		$\{x_2, x_5, x_{10}\}$ $\{x_3, x_5, x_{10}\}$
		$\{x_1, x_6, x_{10}\}$ $\{x_2, x_6, x_{10}\}$
		$\{x_3, x_6, x_{10}\}$ $\{x_1, x_7, x_{10}\}$
		$\{x_2, x_7, x_{10}\}$ $\{x_3, x_7, x_{10}\}$

5. Algebraic application of the operation properties of CRS

Since Pawlak proposed the theory of rough sets in 1982, the researches of algebraic properties on rough sets have been started. However, it is still an open problem regarding the algebraic properties of CRS. As an application on operation properties of CRS, we will investigate some basic algebraic properties of CRS in this section. Firstly, we review some basic concepts of algebraic theory.

A partial order on a nonempty set L is a binary relation $\leq$ such that, for all $x, y, z \in L, (1) x \leq x, (2) x \leq y$ and $y \leq x$ imply $x = y, (3) x \leq y$ and $y \leq z$ imply $x \leq z$. A set L equipped with a partial order is called a partially ordered set. Let $S \subseteq L$, an element $x \in L$ is an upper bound of S if $s \leq x$ for all $s \in S$. Meanwhile, a lower bound can be defined dually. The set of all upper bounds of S is denoted by S^u and the set of all lower bounds of S is denoted by S^l. If S^u has a least element x, then x is called the least upper bound of S. Dually, if S^l has a greatest element x, then x is called the greatest lower bound of S. In what follows, we will denote by $\vee S$ the least upper bound of S and denote by $\wedge S$ the greatest lower bound of S when they exist. In particular, we will write $x \vee y$ in place of $\vee\{x, y\}$ when it exists, and $x \wedge y$ in place of $\wedge\{x, y\}$ when it exists.

Definition 5.1 A partially ordered set L is a lattice, if $a \vee b \in L$ and $a \wedge b \in L$, for all $a, b \in L$.

Definition 5.2 A lattice L is a distribute lattice, if $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$ and $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$, for all $a, b, c \in L$.

Definition 5.3 A distribute lattice $\langle L, \vee, \wedge, \sim\rangle$ is a soft algebra, if for $\forall a, b \in L$, the following three properties are satisfied:

(1) $a \vee 0 = a, a \wedge 0 = 0, a \vee 1 = 1, a \wedge 1 = a$;

(2) $\sim (\sim a) = a$;

(3) $\sim (a \vee b) = (\sim a) \wedge (\sim b), \sim (a \wedge b) = (\sim a) \vee (\sim b)$.

Definition 5.4 A lattice $\langle L, \vee, \wedge, \sim, 0\rangle$ is a pseudo-complement lattice, if for each $a \in L$, there must exist $a^* \in L$ such that a^* is the pseudo-complement element of a. i.e., the following two properties are satisfied:

(1) $a \vee a^* = 0$;

(2) For each $b \in L$, if $a \vee b = 0$, then we have that $b \leq a^*$;

In what follows, based on the operation properties of CRS, we will discuss the algebraic properties of CRS.

5.1. Algebraic properties of the first type of CRS

In this section, we will discuss the algebraic properties of the first type of CRS. Let $(U, \mathcal{C})$ be a minimally unary covering approximation space, and for $\forall x, y \in U$, we have $K_x^{min} = K_y^{min}$ or $K_x^{min} \cap K_y^{min} = \emptyset$. In order to make the discussion more accurate and consistent, we write "$\cup$" in place of "$\vee$", and "$\cap$" in place of "$\wedge$". Then, we have the following conclusions.

Proposition 5.1.1 $(\mathbb{C}^F, \cup, \cap)$ is a lattice.

Proof:
It is clear. □

Proposition 5.1.2 $(\mathbb{C}^F, \cup, \cap)$ is a distribute lattice.

Proof:
For $\forall(\underline{C}(X), \overline{C}(X)), (\underline{C}(Y), \overline{C}(Y)), (\underline{C}(Z), \overline{C}(Z)) \in \mathbb{C}^F$, we have that

$$
\begin{aligned}
&(\underline{C}(X), \overline{C}(X)) \cap ((\underline{C}(Y), \overline{C}(Y)) \cup (\underline{C}(Z), \overline{C}(Z)))\\
=&((\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(Y), \overline{C}(Y))) \cup ((\underline{C}(X), \overline{C}(X)) \cap (\underline{C}(Z), \overline{C}(Z)))\\
&(\underline{C}(X), \overline{C}(X)) \cup ((\underline{C}(Y), \overline{C}(Y)) \cap (\underline{C}(Z), \overline{C}(Z)))\\
=&((\underline{C}(X), \overline{C}(X)) \cup (\underline{C}(Y), \overline{C}(Y))) \cap ((\underline{C}(X), \overline{C}(X)) \cup (\underline{C}(Z), \overline{C}(Z)))
\end{aligned}
$$

Thus the proposition hold. □

Let $(\emptyset, \emptyset) = 0$, $(U, U) = 1$, we have the following conclusions.

Proposition 5.1.3 $(\mathbb{C}^F, \cup, \cap, \sim)$ is a soft algebra.

Proof:
For $\forall(\ \underline{C}(X),\overline{C}(X)\ ),(\ \underline{C}(Y),\overline{C}(Y)\ )\in\mathbb{C}^F$, we have that

(1) $(\ \underline{C}(X),\overline{C}(X)\ )\cup 0=(\ \underline{C}(X),\overline{C}(X)\ )$;

$(\ \underline{C}(X),\overline{C}(X)\ )\cap 0=0$;

$(\ \underline{C}(X),\overline{C}(X)\ )\cup 1=1$;

$(\ \underline{C}(X),\overline{C}(X)\ )\cap 1=(\ \underline{C}(X),\overline{C}(X)\ )$.

(2) $\sim(\sim(\ \underline{C}(X),\overline{C}(X)\ ))=\sim(\sim\overline{C}(X),\sim\underline{C}(X))=(\ \underline{C}(X),\overline{C}(X)\ )$.

(3) $\sim((\ \underline{C}(X),\overline{C}(X)\ )\cap(\ \underline{C}(Y),\overline{C}(Y)\ ))$
$=\sim(\underline{C}(X)\cap\underline{C}(Y),\overline{C}(X)\cap\overline{C}(Y))$
$=(\sim(\overline{C}(X)\cap\overline{C}(Y)),\sim(\underline{C}(X)\cap\underline{C}(Y)))$
$=(\sim(\overline{C}(X))\cup\sim(\overline{C}(Y)),\sim(\underline{C}(X))\cup\sim(\underline{C}(Y)))$
$=(\sim\overline{C}(X),\sim\underline{C}(X))\cup(\sim\overline{C}(Y),\sim\underline{C}(Y))$
$=(\sim(\underline{C}(X),\overline{C}(X))\cup(\sim(\underline{C}(Y),\overline{C}(Y)))$.

Similarly, we have that

$$\sim((\underline{C}(X),\overline{C}(X))\cup(\underline{C}(Y),\overline{C}(Y)))=(\sim(\underline{C}(X),\overline{C}(X))\cap(\sim(\underline{C}(Y),\overline{C}(Y))).$$

Hence, it can be known that $(\ \mathbb{C}^F,\cup,\cap,\sim)$ is a soft algebra. □

For each $(\ \underline{C}(X),\overline{C}(X)\ )\in\mathbb{C}^F$, let $(\ \underline{C}(X),\overline{C}(X)\ )^*=(\ \sim\overline{C}(X),\sim\overline{C}(X)\ )$, then we have the following conclusion.

Proposition 5.1.4 $(\ \mathbb{C}^F,\cup,\cap,\sim,0)$ is a pseudo-complement lattice.

Proof:
For each $(\ \underline{C}(X),\overline{C}(X)\ )\in\mathbb{C}^F$, we have that

(1) $(\ \underline{C}(X),\overline{C}(X)\ )\cap(\ \underline{C}(X),\overline{C}(X)\ )^*=((\ \underline{C}(X),\overline{C}(X)\ )\cap(\ \sim\overline{C}(X),\sim\overline{C}(X)\ ))=(\ \underline{C}(X)\cap(\sim\overline{C}(X)),\overline{C}(X)\cap$
$(\sim\overline{C}(X))\ )=(\emptyset,\emptyset)=0$

(2) For $\forall(\ \underline{C}(X),\overline{C}(X)\ ),(\ \underline{C}(Y),\overline{C}(Y)\ )\in\mathbb{C}^F$, let $(\ \underline{C}(X),\overline{C}(X)\ )\cap(\ \underline{C}(Y),\overline{C}(Y)\ )=0$. Then $(\ \underline{C}(X)\cap\underline{C}(Y),\overline{C}(X)\cap\overline{C}(Y)\ )=0$. We have that $\underline{C}(X)\cap\underline{C}(Y)=\emptyset$. That is to say that $\overline{C}(Y)\subseteq\sim\overline{C}(X)$. Since $\underline{C}(X)\subseteq\overline{C}(X)$, then we have that $\underline{C}(Y)\subseteq\sim\overline{C}(X)$. Hence $(\ \underline{C}(Y),\overline{C}(Y)\ )\subseteq(\ \sim\overline{C}(X),\sim\overline{C}(X)\ ))=(\ \underline{C}(X),\overline{C}(X)\ )^*$. Therefore, $(\ \mathbb{C}^F,\cup,\cap,\sim,0)$ is a pseudo-complement lattice. □

5.2. Algebraic properties of the second type of CRS

In this section, we will study the algebraic properties of the second type of CRS. Let $(U, \mathcal{C})$ be a maximally unary covering approximation space. The proofs of the following propositions can straightforwardly follow by those of propositions listed in Subsection 5.1 and are thus omitted. Hence, we have the following conclusions.

Proposition 5.2.1 $(\mathbb{C}^S, \cup, \cap)$ is a lattice.

Proposition 5.2.2 $(\mathbb{C}^S, \cup, \cap)$ is a distribute lattice.

Proposition 5.2.3 $(\mathbb{C}^S, \cup, \cap, \sim)$ is a soft algebra.

Proposition 5.2.4 $(\mathbb{C}^S, \cup, \cap, \sim, 0)$ is a pseudo-complement lattice.

6. Conclusion

The theory of rough sets based on equivalence relations is an excellent tool to handle granularity of revealing knowledge hidden in information systems. The covering rough set model is an important generalization to the classical rough set model. It is more useful in dealing with uncertainty and granularity. In this paper, we further studied two types of CRS models. We first research the properties of covering approximation operators. To find more excellent results, we study the properties of covering approximation operators with respect to minimally and maximally unary coverings. In addition, we propose the concepts of intersection, union and complement of CRS and study some basic operation properties. Finally, as an application of the operation properties of CRS, some basic algebraic properties of CRS are explored.

There are several issues in covering-based rough sets deserving further research. For example, the results on topological properties of CRS are still fewer. Based on the results listed in this paper, it will be possible to investigate the topological properties of CRS. In addition, there are lots of CRS models based on non-dual approximation operators. In these models, some basic and important operation properties need further study. We will research these issues in the near future.

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